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Statistics of the mesoscopic field

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[1]The measured probability distribution of the microwave field in an ensemble of strongly scattering samples is far from Gaussian. We show, however, that the field in a subensemble with specified total transmission is a Gaussian random variable. This confirms the central hypothesis of random matrix theory of perfect mode mixing and leads to the conclusion that the field and intensity normalized by their respective average magnitudes in a given configuration and the total transmission are statistically independent. This yields a universal form for the intensity correlation function and explains measurements of steady-state and time-resolved transmission in weakly and strongly scattering samples.

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Fluctuations, correlation, and localization of electronic and classical waves in mesoscopic samples have been extensively studied in recent decades [1–3]. The link between these phenomena has been established in the diffusive limit, in the absence of inelastic processes, since all statistical properties may be expressed in terms of a single parameter, the average value of the dimensionless conductance, $g \equiv \langle \Sigma_{ab} |t_{ab}|^2 \rangle$. Here t_{ab} is the transmission coefficient for incident field a into outgoing field b and the sum is carried over all incident and outgoing transverse modes. Localization occurs for $g \sim 1$, but weak localization leads to suppressed transport, whose scaling depends upon g, even in the diffusive limit, $g \gg 1$ [4]. For large g, the probability distribution of total transmission normalized to its ensemble average value, $P(s_a)$, where $s_a = \Sigma_b |t_{ab}|^2 / \langle \Sigma_b |t_{ab}|^2 \rangle$, is given by diagrammatic calculations and random matrix theory (RMT) in terms of g [5,6]. At the same time, the cumulant correlation function of the intensity normalized to its ensemble average for a single polarization component, $C = \langle \delta s_{ab} \delta s_{a'b'} \rangle$, where $s_{ab} = |t_{ab}|^2 / \langle |t_{ab}|^2 \rangle$ and $\delta s_{ab} = s_{ab} - 1$, can be expressed as a perturbation expansion in $1/g$ [7],

$$C = F_{\rm in} F_{\rm out} + \frac{2}{3g}(F_{\rm in} + F_{\rm out}) + \frac{2}{15g^2}(F_{\rm in} F_{\rm out} + F_{\rm in} + F_{\rm out} + 1), \tag{1}$$

where $F_{\rm in}$ and $F_{\rm out}$ represent the square of the field correlation function F_E with respect to change in position or polarization of the source and detector, respectively, $F \equiv |F_E|^2$. Using the isotropic assumption of RMT [8], implying the perfect mode mixing, Kogan and Kaveh [6] showed that in the limit of large number of modes ($N \gg 1$) probability distributions of s_{ab} and s_a are related as follows:

$$P(s_{ab}) = \int_0^\infty ds_a P(s_a) \frac{1}{s_a} \exp\left(-\frac{s_{ab}}{s_a}\right). \tag{2}$$

The distribution $P(s_{ab})$ may be understood as a mixture of negative exponential functions with average intensity s_a with the mixing function $P(s_a)$. A negative exponential intensity distribution results when a large number of statistically independent partial waves are superposed to produce the field at a point; the field distribution in this case would be Gaussian [9]. It is worthwhile therefore to check the fundamental assumption of RMT of the Gaussian field statistics by direct measurements of field statistics.

In the strong scattering limit, correlation is no longer given by Eq. (1), but the correlation function with displacement and polarization shift on the sample output, in diffusive and localized samples, with and without absorption, in both steady state and pulsed transmission measurements can be described by the simple relation [10,11]

$$C = F + \kappa(1 + F), \tag{3}$$

where κ is the degree of correlation corresponding to the value of C, when $F=0$, and equal to $\mathrm{var}(s_a)$. This expression can also be obtained from Eq. (10) of Ref. [12] by replacing the Kronecker δ's by the F functions, for the case in which the position and polarization of the source are not changed, $F_{\rm in}=1$. But the validity of this expression outside the limits of perturbation theory or in the presence of inelastic processes has not been given a theoretical justification. There have recently been measurements of the increasing suppression of pulsed transmission with time delay by weak localization [13], as well as measurements of increasing intensity correlation and fluctuations with time delay [10], but a framework for relating localization and mesoscopic correlation and fluctuations has not been established.

In this Rapid Communication, we show that consideration of the statistics of the mesoscopic field reveal essential connections between fluctuations, correlation, and localization in random media. The field normalized by its average magnitude in a given configuration is shown to be a Gaussian random process. The normalized field and its square ampli-

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tude, the intensity, must therefore be statistically independent of the total transmission. This confirms the fundamental assumption of RMT of the perfect mixing of modes. At the same time, this allows the correlation functions of intensity and field over a random ensemble to be written as products of correlators of the corresponding variables normalized by their average magnitudes in a given configuration, and the correlators, respectively, of the total transmission and its square root. This leads to a simple and universal form for the intensity correlation function with displacement and polarization rotation in terms of the field correlation function and $\mathrm{var}(s_a)$. Finally, the progressive suppression of transmission by weak localization with increasing delay from an exciting pulse is shown to reflect the spectral correlator of the square root of the total transmission.

To examine field statistics, we measure microwave spectra of the field transmission coefficient in ensembles of random quasi-one-dimensional samples of alumina spheres with the use of a vector network analyzer. Alumina spheres with diameter 0.95 cm and refractive index 3.14 are embedded in Styrofoam spheres of refractive index 1.04 to produce an alumina volume fraction of 0.068. The spheres are contained in a copper tube of diameter 7.3 cm and length 61 cm with reflecting sidewalls and open ends. Measurements are made in 10^4 sample configurations produced by rotating the sample tube. Spectra are taken in steps of 0.3 MHz in the frequency intervals 14.7–15.7 GHz (measurement A) and 9.95–10.15 GHz (measurement B), in which waves are diffusive and localized, respectively [14]. In the limit of Gaussian field statistics for the random ensemble, $\mathrm{var}(s_{ab})=1$ [9]. In measurements A and B, $\mathrm{var}(s_{ab})=1.18$ and 6.18, respectively. In order to study field statistics in even more strongly correlated samples, we examine the statistics of pulsed transmission of localized waves at a long delay from an exciting pulse, since the degree of correlation increases with time delay [10,15]. The response to a pulse with a Gaussian temporal envelope of width $\sigma_t=160$ ns is obtained from the Fourier transform of the product of the field transmission spectra of measurement B and a Gaussian spectral function of bandwidth $\sigma=1$ MHz$\approx 0.6\delta\nu$, where $\delta\nu$ is the field correlation frequency. The field statistics are examined for waves delayed by 740 ns from the center of exciting pulse. In this case (measurement C), $\mathrm{var}[s_{ab}(t)]=20.1$.

We first consider the probability distribution of the field transmission coefficient normalized to its ensemble average magnitude, $E=t_{ab}/\sqrt{\langle|t_{ab}|^2\rangle}$. We find in all cases that the statistics of the real and imaginary parts of the field, $r=\mathrm{Re}[E]$ and $i=\mathrm{Im}[E]$, respectively, are the same and that r and i have zero mean and vanishing cross correlation, $\langle r\times i\rangle=0$. The distributions $P(\alpha)$, where $\alpha=r$, i, are shown by the solid curves in Fig. 1. Increasing deviations from a Gaussian distribution, $P(\alpha)=(1/\sqrt{\pi})\exp(-\alpha^2)$, obtained for the model of the field as a random phasor sum (dashed curve) [9], are seen for the distributions in measurements with increasing values of $\mathrm{var}(s_{ab})$.

From the RMT formalism [8], the field transmission coefficient is given by $t_{ab}=\Sigma_k u_{ak}\sqrt{\tau_k}v_{kb}$, where u_{ak} and v_{kb} are matrix elements of unitary matrices u and v, respectively, and $\{\tau_k\}$ is a set of N transmission eigenvalues. In the isotropic approximation, in the limit, $N\gg 1, u_{ak}$, and v_{kb} are circular Gaussian variables, statistically independent of each other and of $\{\tau_k\}$ [8,16]. In a given configuration, for a fixed incident mode a, t_{ab} is a sum of jointly Gaussian random variables and is, therefore, itself Gaussian [9]. Therefore, the conditional probability distribution of the real and imaginary parts of the field given total transmission s_a is

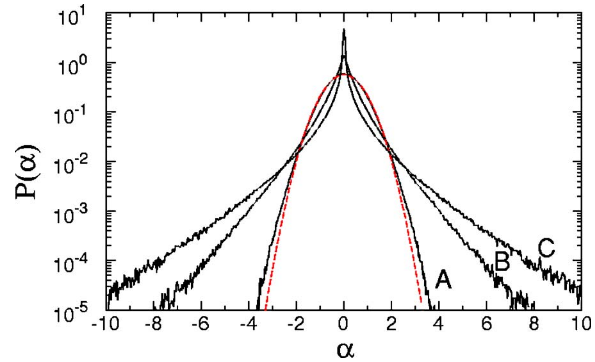

FIG. 1. (Color online) Field distributions $P(\alpha)$ of measurements A,B, and C (solid). The dashed curve is the Gaussian distribution, $P(\alpha)=(1/\sqrt{\pi})\exp(-\alpha^2)$, of the model of the field as a random phasor sum (Ref. [9]).

$$P(r,i|s_a)=\frac{1}{\pi s_a}\exp\left(-\frac{r^2+i^2}{s_a}\right),$$

and the full distribution $P(r,i)$ is given by

$$P(r,i)=\int_0^\infty ds_a P(s_a)\frac{1}{\pi s_a}\exp\left(-\frac{r^2+i^2}{s_a}\right). \tag{4}$$

Since $s_{ab}=|E|^2=r^2+i^2$, it then follows from Eq. (4) that the intensity distribution $P(s_{ab})$ is given by Eq. (2).

When the field E is normalized by $\sqrt{s_a}$ in a given configuration, $E'=E/\sqrt{s_a}$, the conditional distribution of the normalized field, $P(i',r'|s_a)$, is independent of s_a, $P(i',r'|s_a)=P(i',r')=(1/\pi)\exp[-(r'^2+i'^2)]$, where $r'=\mathrm{Re}[E']$ and $i'=\mathrm{Im}[E']$. This implies that the joint probability distribution function $P(i',r';s_a)$ factors into a product of two distributions, $P(i',r';s_a)=P(r',i')\times P(s_a)$, or that E' and s_a are statistically independent. It then follows that the field E can be expressed as a product of two statistically independent variables, $E=E'\times\sqrt{s_a}$. The intensity s_{ab} can also be written as a product of two statistically independent variables, $s_{ab}=s'_{ab}\times s_a$, where $s'_{ab}=|E'|^2=r'^2+i'^2$. The corresponding joint distribution is $P(s'_{ab};s_a)=P(s'_{ab})\times P(s_a)$, where $P(s'_{ab})=\exp(-s'_{ab})$. Since the nth moment of the product of two statistically independent random variables is the product of their nth moments, the moment relation between the field and the total transmission is

$$\langle\alpha^n\rangle=\langle\alpha'^n\rangle\langle s_a^{n/2}\rangle=\begin{cases}\frac{(2k-1)!!}{2^k}\langle s_a^k\rangle, & n=2k\\ 0, & n=2k-1,\end{cases} \tag{5}$$

where $\alpha'=r'$, i' and k is an integer, and the moment relation between the intensity and the total transmission is

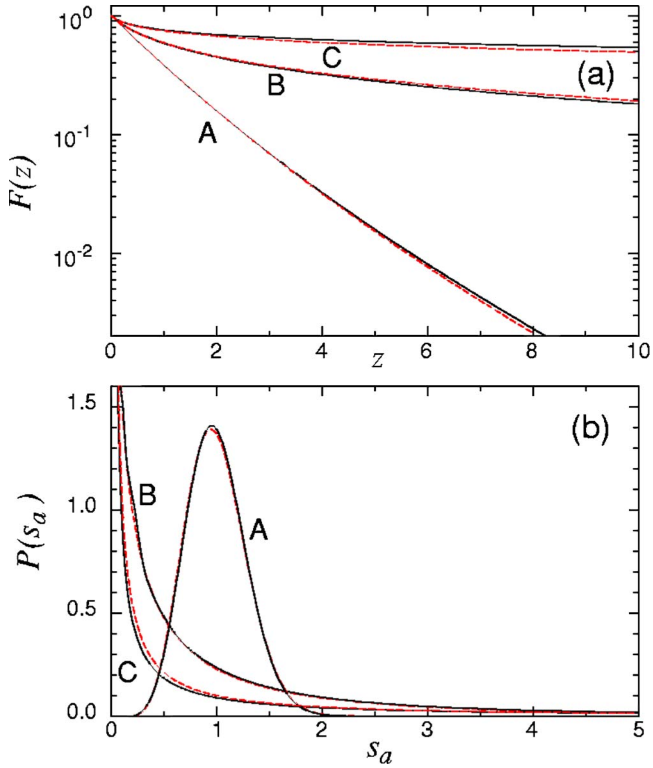

FIG. 2. (Color online) (a) Laplace transform $F(z)$ of the total transmission distribution $P(s_a)$ found as the field average, $F(z) = \langle \cos(2\alpha\sqrt{z})\rangle$, from measurements A,B, and C (solid). The dashed curves are $F(z)$ of Eq. (7) (Refs. [5] and [6]) with g replaced by $2/3\mathrm{var}(s_a)$. (b) $P(s_a)$ found by inverting the corresponding $F(z)$ in (a). For measurement $A, F(z)$ is fit by $F(z)=\exp[Q(z)]$, where $Q(z)$ is a polynomial of power 3, and the resulting function is inverted using the Weeks method (Ref. [20]); for measurements B and $C, F(z)$ is inverted using the Stehfest method (Ref. [21]).

$$\langle s_{ab}^n\rangle = \langle s'^n_{ab}\rangle\langle s_a^n\rangle = n!\,\langle s_a^n\rangle. \tag{6}$$

Equation (6) is in agreement with calculations of Ref. [6] and measurements of Ref. [19].

To check whether $P(r,i)$ is indeed the mixture of Gaussian distributions of Eq. (4), we solve for $P(s_a)$ by utilizing the field average $\langle\cos(2\alpha\sqrt{z})\rangle$, with $z \geqslant 0$. Expanding the cosine as a power series in α and then using the moment relation of Eq. (5), we obtain the Laplace transform of $P(s_a), F(z)$, $\langle\cos(2\alpha\sqrt{z})\rangle = \langle\exp(-s_a z)\rangle \equiv F(z)$. The solid curves in Fig. 2(a) are plots of $F(z)=\langle\cos(2\alpha\sqrt{z})\rangle$ vs z for measurements A,B, and C. These curves are then inverted to obtain $P(s_a)$, by using the approximate inversion of the Laplace transform [20,21]. The distributions $P(s_a)$ are shown by the solid curves in Fig. 2(b). This confirms the isotropic assumption of RMT and that the field distribution in an ensemble of mesoscopic samples is the mixture of Gaussian distributions given by Eq. (4).

$F(z)$ was previously found [5,6] in the diffusive limit, $g \gg 1$, in the absence of absorption, to be a function only of the dimensionless conductance g,

$$F(z) = \exp[-g\ln^2(\sqrt{1+z/g} + \sqrt{z/g})]. \tag{7}$$

A nonperturbative result for $F(z)$ was obtained in Ref. [22] for the case of broken time-reversal symmetry. Theoretical expressions for $P(s_a)$ and $P(s_{ab})$ derived using Eq. (7) were found to closely match the measured distributions [19], even at the localization threshold, $g \sim 1$, and even in the presence of absorption, once g was replaced by $2/3\mathrm{var}(s_a)$. $\mathrm{Var}(s_a)$ can be found from the relation, $\mathrm{var}(s_a)=\frac{1}{2}[\mathrm{var}(s_{ab})-1]$, following from Eq. (6). The plots of $F(z)$ of Eq. (7) with g replaced by $2/3\mathrm{var}(s_a)$ with the values of $\mathrm{var}(s_a)$ of 0.09, 2.59, and 9.55 for measurements A,B, and C, respectively, are displayed as the dashed curves in Fig. 2(a). The theoretical curves are seen to deviate from $F(z)$ found at large z. Deviations also appear in $P(s_a)$ for $s_a<1$ [Fig. 2(b)]; increasing deviations from the approximate theory of Refs. [5] and [6] for $P(s_a)$ are observed for the distributions with larger values of $\mathrm{var}(s_a)$.

We now consider field and intensity correlation versus shifts in the position and polarization of the detected wave, p, for which the total transmission does not change, and in the incident frequency or positions of scatterers within the medium, q, which engenders a change in the total transmission. By virtue of the stationarity and statistical independence of the normalized field $E'(p,q)$ and the total transmission $s_a(q)$, the field correlation function with shifts in p and q, Δp and Δq, $F_E(\Delta p,\Delta q)=\langle E(p,q)E^*(p+\Delta p,q+\Delta q)\rangle$, can be written as the product of two correlation functions

$$F_E(\Delta p,\Delta q) = F_{E'}(\Delta p,\Delta q) \times \Gamma_{\sqrt{s_a}}(\Delta q), \tag{8}$$

where $F_{E'}(\Delta p,\Delta q)=\langle E'(p,q)E'^*(p+\Delta p,q+\Delta q)\rangle$ and $\Gamma_{\sqrt{s_a}}(\Delta q)=\langle\sqrt{s_a(q)s_a(q+\Delta q)}\rangle$. Similarly, the intensity correlation function, $\Gamma_{s_{ab}}(\Delta p,\Delta q)=\langle s_{ab}(p,q)s_{ab}(p+\Delta p,q+\Delta q)\rangle$, can be written as

$$\Gamma_{s_{ab}}(\Delta p,\Delta q) = \Gamma_{s'_{ab}}(\Delta p,\Delta q) \times \Gamma_{s_a}(\Delta q), \tag{9}$$

where $\Gamma_{s'_{ab}}(\Delta p,\Delta q)=\langle s'_{ab}(p,q)s'_{ab}(p+\Delta p,q+\Delta q)\rangle$ and $\Gamma_{s_a}(\Delta q)=\langle s_a(q)s_a(q+\Delta q)\rangle$. The Siegert relation for the Gaussian random process $E'(p,q)$ gives $\Gamma_{s'_{ab}}(\Delta p,\Delta q)=1+F'(\Delta p,\Delta q)$, where $F'(\Delta p,\Delta q) = |F_{E'}(\Delta p,\Delta q)|^2$ [9].

For a shift in position or polarization, Δp, with $\Delta q=0$, $\Gamma_{\sqrt{s_a}}(\Delta q)=\langle s_a\rangle=1$, and $\Gamma_{s_a}(\Delta q)=\langle s_a^2\rangle$. This gives $F_E(\Delta p) = F_{E'}(\Delta p)$ and $\Gamma_{s_{ab}}(\Delta p)=\langle s_a^2\rangle[1+F'(\Delta p)]$. Then, the cumulant intensity correlation function is given by $C(\Delta p) = \Gamma_{s_{ab}}(\Delta \mathrm{p})-1=F'(\Delta \mathrm{p})+\mathrm{var}(s_a)[1+F'(\Delta \mathrm{p})]$, in agreement with Eq. (3). This confirms that the field normalized by its average magnitude in a given configuration, $E'(p,q)$, is a Gaussian random process. For a shift in position, $\Delta p=\Delta r$, the functional form of $F_{E'}(\Delta r)$ is that predicted by coherence theory [17,18], as found in microwave [10] and numerical [11] studies of field correlation. For correlation with polarization rotation of the detected field, $\Delta p=\Delta\theta$, $F_{E'}(\Delta\theta) = \cos(\Delta\theta)$ [10].

When the total transmission s_a varies with the incident frequency ν, or with time τ as the internal structure of the sample changes, mesoscopic correlation can no longer be expressed through the single parameter, $\mathrm{var}(s_a)$. For example, the field and intensity correlation functions with frequency shift are $F_E(\Delta\nu)=F_{E'}(\Delta\nu)\times\Gamma_{\sqrt{s_a}}(\Delta\nu)$ and $\Gamma_{s_{ab}}(\Delta\nu)=[1+F'(\Delta\nu)]\times\Gamma_{s_a}(\Delta\nu)$, respectively. In addition, the correlation function $\Gamma_{\sqrt{s_a}}(\Delta\nu)$ is unity at $\Delta\nu=0$ and falls to a value $\langle\sqrt{s_a}\rangle^2$ at large values of $\Delta\nu$. In the limit of large frequency shifts, $F_E(\Delta\nu)=\langle\sqrt{s_a}\rangle^2 F_{E'}(\Delta\nu)$, which is consistent with results of Ref. [23]. The field correlation function $F_E(\Delta\nu)$ is of interest because it is the Fourier transform of the time-of-flight distribution, $P(t)$, where t is the time delay following a short pulse [24]. Since $F_E(\Delta\nu)$ is the product of two functions, $P(t)$ is the convolution of their two Fourier transforms. One therefore expects that $P(t)$ is the sum of two terms. The first term is associated with spectral correlation of the Gaussian field E', and the second is due to the decorrelation of the square root of total transmission $\sqrt{s_a}$.

In conclusion, the transmitted field normalized by its average magnitude in same sample configuration is a Gaussian random process with position, polarization, frequency, and time. This confirms the fundamental assumption of the RMT of the perfect mode mixing and leads to simple expressions for the field and intensity correlation functions in mesoscopic samples. The field correlation function with displacement and polarization rotation is independent of closeness to the localization threshold or of the degree of correlation, κ, while the intensity correlation function is given in terms of the field correlation function and $\kappa=\mathrm{var}(s_a)$. In contrast, the field correlation function with frequency or time shift is written as a product of correlators of the Gaussian field and $\sqrt{s_a}$. Since the time-of-flight distribution for particles is the Fourier transform of the field correlation function with frequency shift, the increasing suppression of transport with time delay due to weak localization is associated with mesoscopic fluctuations.

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